Theorem 3.19

If $S = \{\vec{V}_1, \vec{V}_2, .., \vec{V}_k\}$ are vectors in $\mathbb{R}^n$ then span(S) is a Subspace of $\mathbb{R}^n$.

Examples: a) Lines throug the origin in $\mathbb{R}^3$.

$$S = \left\{ \vec{X} \text{ in } \mathbb{R}^3 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ t \text{ in } \mathbb{R} \right\}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b) Planes in $\mathbb{R}^3$, containg the origin.

$$S = \left\{ \vec{X} \text{ in } \mathbb{R}^3 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + t \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \right\}$$

Where $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ are lin. indep.

Proof of Theorem 3.19

1) If $\vec{u}$ is in S and $\vec{w}$ is in S,

then

$$\vec{u}+\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$
$$+ d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$$
$$= (c_1+d_1)\vec{v}_1 + \cdots + (c_k+d_k)\vec{v}_k$$
$$= f_1\vec{v}_1 + \cdots + f_k\vec{v}_k,$$

which implies

$\vec{u}+\vec{v}$ is in S.

2) If c is an arbitrary scalar

$$c\vec{u} = c(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k)$$
$$= cc_1\vec{v}_1 + cc_2\vec{v}_2 + \cdots + cc_k\vec{v}_k$$
$$= h_1\vec{v}_1 + h_2\vec{v}_2 + \cdots + h_k\vec{v}_k,$$

which implies

$c\vec{u}$ is in S.

3) $\vec{0}$ is in S, because we can choose

a vector $\vec{u}$ in S, which is obtained by choosing all scalars $c_i = 0$. It means

$$0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k = \vec{0}.$$

Consider the matrix $A_{3\times 5}$

$$A = \begin{array}{c} \begin{array}{ccccc} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 & \vec{u}_5 \end{array} \\ \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \\ -3 & 6 & -1 & 1 & -7 \end{bmatrix} \end{array} \begin{array}{l} \\ \vec{R}_1 \to \\ \vec{R}_2 \to \\ \vec{R}_3 \to \end{array}$$

Two important Subspaces associated to A are

1) Row space of A : $\boxed{\text{row}(A)}$

It consists of all vectors in

$\boxed{\text{Span}(\vec{R}_1, \vec{R}_2, \vec{R}_3)}$ Span of row vectors of A

$\text{row}(A) \subset \mathbb{R}^5$

contained in

2) Column space of A : $\boxed{\text{Col}(A)}$

It consists of all vectors in

$\boxed{\text{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5)}$ Span of Column vectors of A

$\text{Col}(A) \subset \mathbb{R}^3$

Subspace: Null Space of A or null(A)

Thm 3.21 $\quad A_{m\times n}$ matrix

$$N = \left\{ \boxed{\vec{x}} : \boxed{A\vec{x} = \vec{0},} \ \vec{x} \text{ in } \mathbb{R}^n \right\}$$

Then N is a subspace of $\mathbb{R}^n$.

Proof.

i) $\vec{0}$ is in N

Because, if $\vec{x} = \vec{0}$, then $A\vec{x} = A\vec{0} = \vec{0}$

then $\vec{0}$ is in N.

ii) If $\vec{u}$ and $\vec{v}$ are in N, then $\vec{u}+\vec{v}$ is in N.

w.t.p. $\quad A(\vec{u}+\vec{v}) = \vec{0}$

we know: $\quad A\vec{u} = \vec{0}, \quad A\vec{v} = \vec{0}.$

then $\quad A(\vec{u}+\vec{v}) \xlongequal[\text{operations}]{\text{Matrix}} A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$ ✓

So $\vec{u}+\vec{v}$ is in N.

iii) If c is in $\mathbb{R}$ (scalar) and u is in N

then $c\vec{u}$ is in N. (i.e., $A(c\vec{u}) \overset{\text{w.t.p.}}{=} \vec{0}$)

Proof. $\quad A(c\vec{u}) \xlongequal[\text{Operations}]{\text{Matrix}} c(A\vec{u}) = c\vec{0} = \vec{0}.$ ✓

Then $c\vec{u}$ is in N.

Constructing bases for: row(A), col(A), and null(A)

An important step in the construction of bases for these subspaces consists of reducing the matrix A to an echelon form:

$$A = \begin{array}{c c} \begin{array}{ccccc} \vec{u}_1\downarrow & \vec{u}_2\downarrow & \vec{u}_3\downarrow & \vec{u}_4\downarrow & \vec{u}_5\downarrow \end{array} & \\ \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \\ -3 & 6 & -1 & 1 & -7 \end{bmatrix} & \begin{array}{l} \vec{R}_1 \rightarrow \\ \vec{R}_2 \rightarrow \\ \vec{R}_3 \rightarrow \end{array} \end{array}$$

$$A' = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 5 & 10 & -10 \end{bmatrix} \begin{array}{l} \vec{R}_1' = \vec{R}_1 \\ \vec{R}_2' = \vec{R}_2 - 2\vec{R}_1 \\ \vec{R}_3' = \vec{R}_3 + 3\vec{R}_1 \end{array}$$

$$A'' = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \boxed{\vec{R}_1'' = \vec{R}_1' = \vec{R}_1} \\ \boxed{\vec{R}_2'' = \vec{R}_2' = \vec{R}_2 - 2\vec{R}_1} \\ \boxed{\vec{R}_3'' = \vec{R}_3' - 5\vec{R}_2'} \end{array}$$

$$\vec{R}_3'' = \vec{R}_3' - 5\vec{R}_2' = \vec{R}_3 + 3\vec{R}_1 - 5(\vec{R}_2 - 2\vec{R}_1)$$

or $\boxed{\vec{R}_3'' = \vec{R}_3 + 13\vec{R}_1 - 5\vec{R}_2}$

$\boxed{\vec{R}_1'' = \vec{R}_1}$ and $\boxed{\vec{R}_2'' = \vec{R}_2 - 2\vec{R}_1}$

We want to know how the two Subspaces

$row(A) = Span(\vec{R}_1, \vec{R}_2, \vec{R}_3)$ and $row(A'') = Span(\vec{R}_1'', \vec{R}_2'', \vec{R}_3'')$

are related.

we know

$\boxed{\vec{R}_1'' = R_1}$, $\boxed{R_2'' = R_2 - 2R_1}$

$\boxed{R_3'' = \vec{R}_3 + 13\vec{R}_1 - 5\vec{R}_2}$

Then, if

$\vec{w}$ is in $row(A'')$

$$\vec{w} = c_1\vec{R}_1'' + c_2\vec{R}_2'' + c_3R_3''$$

$$= c_1\vec{R}_1 + c_2(\vec{R}_2 - 2\vec{R}_1) + c_3(\vec{R}_3 + 13\vec{R}_1 - 5\vec{R}_2)$$

$$= (c_1 + 13c_3)\vec{R}_1 + (c_2 - 5c_3)\vec{R}_2 + c_3\vec{R}_3$$

Then

$\vec{w}$ is in $Span(\vec{R}_1, \vec{R}_2, \vec{R}_3) = row(A)$.

or

$$\boxed{row(A'') \subset row(A)}$$

by reversing the operation (matrix operations)

we also get $row(A) \subset row(A'')$

Then $\boxed{row(A) = row(A'')}$

From previous analysis

$$\mathrm{Span}\,(R_1'', R_2'', \overset{=\vec{0}}{R_3'''}) = \mathrm{Span}\,(R_1'', R_2'')$$
$$= \mathrm{row}\,(A)$$

So if they are also linearly indep.

$$B_{\mathrm{row}(A)} = \{\vec{R}_1'', R_2''\} \text{ is a basis for } \mathrm{row}(A).$$

Now, we know from simple inspection that these two vectors are linearly independent.

$$R_1'' = [1\ -2\ 2\ 3\ -1],\quad R_2'' = [0\ 0\ 1\ 2, -2].$$

Therefore,

$\cdot\, B_{\mathrm{row}(A)}$ is a basis for row (A).

– Practical value of this analysis is that we can get a basis for A by choosing the linearly independent vectors (all) of an echelon form of A.

How about Column A or Col(A).

Reconsider

$$A = \begin{array}{c} \begin{array}{ccccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \vec{c}_4 & \vec{c}_5 \end{array} \\ \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \\ -3 & 6 & -1 & 1 & -7 \end{bmatrix} \end{array} \sim \begin{array}{c} \begin{array}{ccccc} \vec{f}_1 & \vec{f}_2 & \vec{f}_3 & \vec{f}_4 & \vec{f}_5 \end{array} \\ \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = B$$

Obviously, $Col(A) \neq Col(B)$, why?

Also, notice that

$$\boxed{\begin{array}{c} \vec{f}_2 = -2\vec{f}_1, \quad \vec{f}_4 = 2\vec{f}_3 - \vec{f}_1 \\ \vec{f}_5 = -2\vec{f}_3 + 3\vec{f}_1. \end{array}} \qquad (8.1)$$

and -

$$\boxed{\vec{C}_2 = -2\vec{C}_1, \quad \vec{C}_4 = 2\vec{C}_3 - \vec{C}_1 \\ C_5 = -2\vec{C}_3 + 3\vec{C}_1.} \qquad (9.1)$$

Therefore,

$$Col(A) = span(\vec{C}_1, \vec{C}_2, \vec{C}_3, \vec{C}_4, \vec{C}_5) = span(\vec{C}_1, \vec{C}_3)$$

Equs. (8.1) and (9.1) implies that the columns of A have the same <u>dependence</u> relations

as the dependence among the columns of B.

As a consequence,

$\vec{C}_1$ and $\vec{C}_3$ are lin. indep., because $\vec{f}_1$ and $\vec{f}_3$ are lin. indep.

Summarizing, the set

$B_2 = \{\vec{C}_1, \vec{C}_3 \text{ in } \mathbb{R}^3\}$ is lin. indep. and span Col(A).

So B_2 is a basis for col(A).

Constructing a matrix for null(A).

Find $\vec{x}$ such that $A\vec{x} = \vec{0}$.

Augmented matrices:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \\ -3 & 6 & -11 & 1 & -7 & 0 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then, $\boxed{x_3 = 2x_5 - 2x_4}$

$$x_1 = x_5 - 3x_4 - 2x_3 + 2x_2$$

or $x_1 = x_5 - 3x_4 - 2(2x_5 - 2x_4) + 2x_2$

$$\boxed{x_1 = -3x_5 + x_4 + 2x_2}$$

Then, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}$ is in null(A) if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_5 + x_4 + 2x_2 \\ x_2 \\ 2x_5 - 2x_4 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \overset{\vec{g}_1}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \overset{\vec{g}_2}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + x_5 \overset{\vec{g}_3}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

Then, $\text{null}(A) = \text{Span}(\vec{g}_1, \vec{g}_2, \vec{g}_3)$

Also $\vec{g}_1, \vec{g}_2, \vec{g}_3$ are lin. indep. (why?)

Thus, the set $G = \{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ is a basis for null(A).

Summarizing for $A_{m\times n}$

Thm 3.20 If $A \sim B \Rightarrow \text{row}(A) = \text{row}(B)$.
(Explain)

Not true for col(A) and col(B)., but . . .

Thm 3.21 1) $A_{m\times n}$
2) N is set of Vectors Solns. of
$$A\vec{x} = \vec{0}.$$
then, N is a Subspace

Proof.- (Easy).

Def. $A_{m\times n}$.

Null space of A = null(A) =
$$= \{\vec{x} \text{ in } \mathbb{R}^n : \quad A\vec{x} = \vec{0}\}$$

Example: Find a basis for null(A) for the previous matrix A.

the strategy is find a representation of any Solution. by Solving the System corresponding to the augmented matrix
$$[A|\vec{0}].$$

Dimension

Thm 3.23 Any two bases for a Subspace S of $\mathbb{R}^n$ have the same number of Vectors.

Def. The number of Vectors in a basis for S is called the dimension of S (Subspace of $\mathbb{R}^n$). It's denoted as

$$\dim S$$

Examples:

a) $\dim \{\vec{0}\} = 0$, why?

$$\left.\begin{array}{l} \text{b) } \dim \text{row}(A) = 2 \\ \text{c) } \dim \text{Col}(A) = 2 \\ \text{d) } \dim \text{null}(A) = 3 \end{array}\right\} \text{for our previous matrix } A.$$

e) $\dim \mathbb{R}^n = n$

Notice that

1) $\dim(\text{row}(A)) = \dim(\text{Col}(A))$

$$\underset{\substack{\#\text{ nonzero rows}\\ \text{of echelon}\\ \text{form}}}{\parallel} \qquad = \qquad \underset{\substack{\#\text{ of Columns of echelon form of } A\\ \text{with a pivot position}}}{\parallel}$$

<u>Thm 3.24</u> The row and column spaces of A have the same dimension.

<u>Def.</u> The <u>rank of a matrix A</u> is the common number

$$\underline{\mathrm{rank}(A)} = \dim(\mathrm{row}(A)) = \dim(\mathrm{col}(A))$$

This concept was introduced earlier as the # of nonzero rows in an echelon form of A.

Notice, that $\mathrm{row}(A)$ is a subspace of $\mathbb{R}^n$
and $\mathrm{Col}(A)$ is a Subspace of $\mathbb{R}^m$
for $A_{m\times n}$.

<u>Thm 3.25</u> For any matrix A

$$\mathrm{rank}(A^T) = \mathrm{rank}(A).$$

<u>Proof.</u>

$\mathrm{rank}(A^T) =$

Def.- The nullity of a matrix A is the dimension of its null space and is denoted as

$$\text{nullity}(A).$$

Observe that

$$\text{nullity}(A) = \#\text{ free variables in the solution.}$$

Reformulation of rank theorem (Thm 3.26)

If $A_{m \times n}$, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Proof. (easy).

Theorem 3.27 **The Fundamental Theorem of Invertible Matrices: Version 2**

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.
f. $\text{rank}(A) = n$
g. $\text{nullity}(A) = 0$
h. The column vectors of A are linearly independent.
i. The column vectors of A span $\mathbb{R}^n$.
j. The column vectors of A form a basis for $\mathbb{R}^n$.
k. The row vectors of A are linearly independent.
l. The row vectors of A span $\mathbb{R}^n$.
m. The row vectors of A form a basis for $\mathbb{R}^n$.

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The nullity of a matrix was defined in 1884 by James Joseph Sylvester (1814–1887), who was interested in *invariants*—properties of matrices that do not change under certain types of transformations. Born in England, Sylvester became the second president of the London Mathematical Society. In 1878, while teaching at Johns Hopkins University in Baltimore, he founded the *American Journal of Mathematics*, the first mathematical journal in the United States.

Proof We have already established the equivalence of (a) through (e). It remain[s to] be shown that statements (f) to (m) are equivalent to the first five statements.

(f) $\Leftrightarrow$ (g) Since $\text{rank}(A) + \text{nullity}(A) = n$ when A is an $n \times n$ matrix, it follows fr[om] the Rank Theorem that $\text{rank}(A) = n$ if and only if $\text{nullity}(A) = 0$.

(f) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (h) If $\text{rank}(A) = n$, then the reduced row echelon form of A ha[s] n leading 1s and so is I_n. From (d) $\Rightarrow$ (c) we know that $A\mathbf{x} = \mathbf{0}$ has only the tri[vial] solution, which implies that the column vectors of A are linearly independent, si[nce] $A\mathbf{x}$ is just a linear combination of the column vectors of A.

(h) $\Rightarrow$ (i) If the column vectors of A are linearly independent, then $A\mathbf{x} = \mathbf{0}$ has on[ly] the trivial solution. Thus, by (c) $\Rightarrow$ (b), $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$. This means that every vector $\mathbf{b}$ in $\mathbb{R}^n$ can be written as a linear combination of the column vectors of A, establishing (i).

(i) $\Rightarrow$ (j) If the column vectors of A span $\mathbb{R}^n$, then $\text{col}(A) = \mathbb{R}^n$ by definit[ion], so $\text{rank}(A) = \dim(\text{col}(A)) = n$. This is (f), and we have already established th[at] (f) $\Rightarrow$ (h). We conclude that the column vectors of A are linearly independent an[d so] form a basis for $\mathbb{R}^n$, since, by assumption, they also span $\mathbb{R}^n$.

(j) $\Rightarrow$ (f) If the column vectors of A form a basis for $\mathbb{R}^n$, then, in particular, they a[re] linearly independent. It follows that the reduced row echelon form of A contain[s n] leading 1s, and thus $\text{rank}(A) = n$.

The above discussion shows that (f) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (h) $\Rightarrow$ (i) $\Rightarrow$ (j) $\Rightarrow$ (f) $\Leftrightarrow$ (g). Now recall that, by Theorem 3.25, $\text{rank}(A^T) = \text{rank}(A)$, so what we ha[ve] just proved gives us the corresponding results about the column vectors of A^T. Th[ese] are then results about the *row* vectors of A, bringing (k), (l), and (m) into the networ[k of] equivalences and completing the proof.

Theorems such as the Fundamental Theorem are not merely of theoretical int[er]est. They are tremendous labor-saving devices as well. The Fundamental Theor[em] has already allowed us to cut in half the work needed to check that two square ma[tri]ces are inverses. It also simplifies the task of showing that certain sets of vectors [are] bases for $\mathbb{R}^n$. Indeed, when we have a set of n vectors in $\mathbb{R}^n$, that set will be a basis [for] $\mathbb{R}^n$ if *either* of the necessary properties of linear independence or spanning set is tr[ue]. The next example shows how easy the calculations can be.

Coordinates

Thm 3.29 For S a subspace of $\mathbb{R}^n$

If $B = \{\vec{v}_1, .., \vec{v}_k\}$ is a basis for S, then every vector $\vec{v}$ in S can be written as

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

and the coefficients $c_1.., c_k$ are unique.

Proof. (Do it!).

Def. The coefficients $c_1.., c_k$ of the previous thm are called the coordinates of $\vec{v}$ with respect to B, and the column vector

$$[\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$ is called the coord. vector of $\vec{v}$ with respect to B.